Identificar campos gradientes.

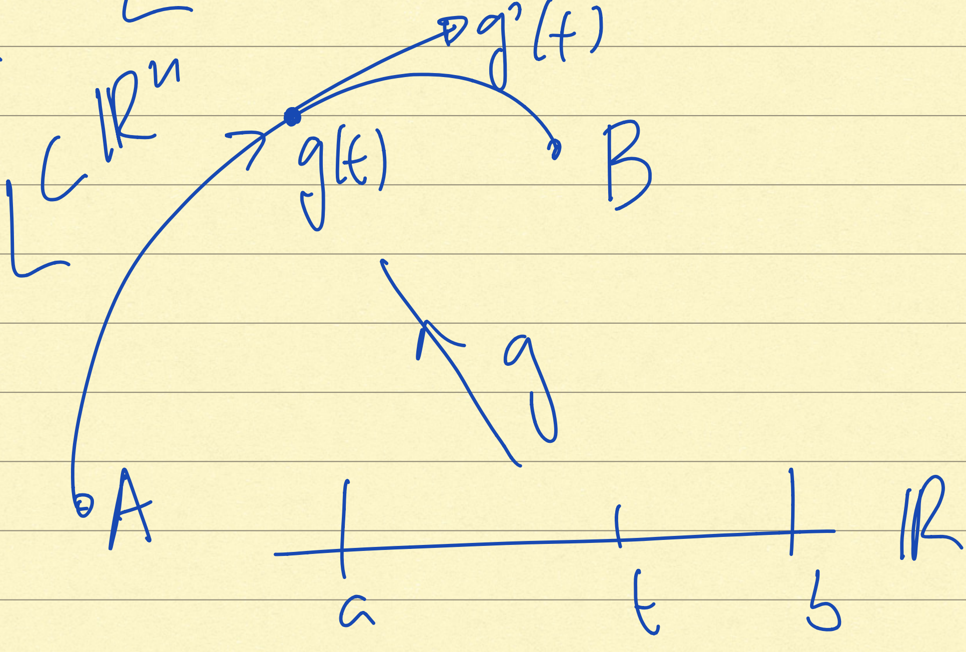
$F : \mathbb{R}^n \to \mathbb{R}^n$, $\boxed{F = \nabla \varphi}$

$\varphi : \mathbb{R}^n \to \mathbb{R}$, C^1. $\longrightarrow$ Potencial escalar de F.

Teorema Fundamental do Cálculo:

$\boxed{\text{Se } F = \nabla\varphi}$, então

$$\int_L F \cdot dg = \varphi(B) - \varphi(A)$$



Se $\varphi \in C^2$, $\boxed{F = \nabla \varphi}$, então

$$(F_1, F_2, \dots, F_n) = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \dots, \frac{\partial \varphi}{\partial x_n}\right)$$

$$F_j = \frac{\partial \varphi}{\partial x_j} \qquad j = 1, \dots, n$$

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial \varphi}{\partial x_j}\right) = \frac{\partial^2 \varphi}{\partial x_k \partial x_j}$$

$$= \frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\left(\frac{\partial \varphi}{\partial x_k}\right)$$

$$\boxed{\frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j}} \qquad j \neq k$$

Condição necessária para que F seja gradiente.

Definição: Se $\dfrac{\partial F_k}{\partial x_j} = \dfrac{\partial F_j}{\partial x_k}, j \neq k$

diz-se que F é um campo vectorial FECHADO.

$\mathbb{R}^2$: $F: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$

$\mathbb{R}^3$: $F: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{cases} \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} \\ \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x} \\ \dfrac{\partial F_2}{\partial z} = \dfrac{\partial F_3}{\partial y} \end{cases}$$

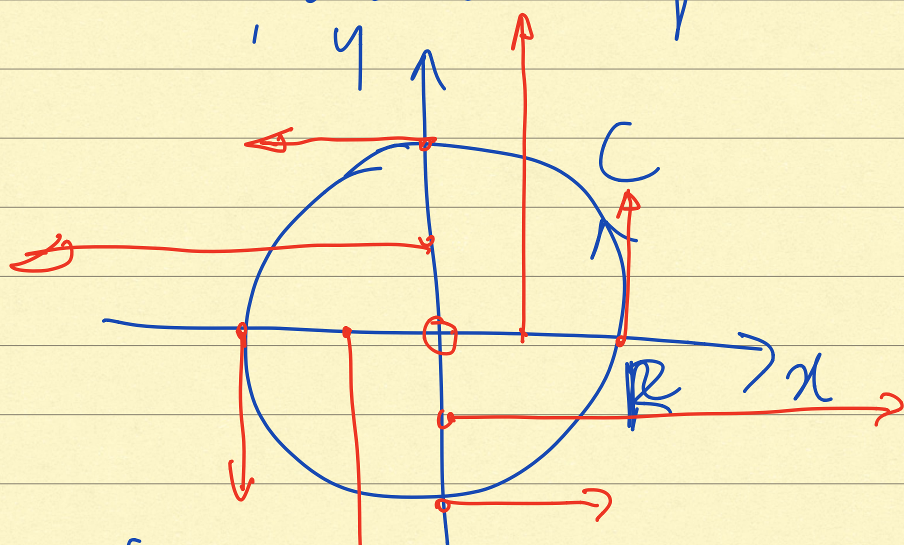
Exemplo: $F(x,y) = \left(-\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$

$$(x,y) \neq \boxed{(0,0)}$$

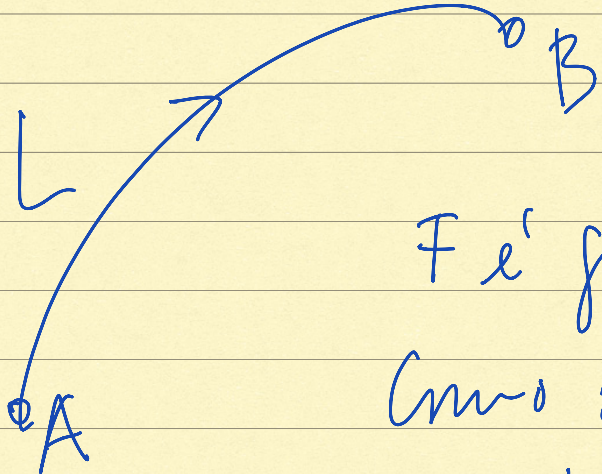
Exercício: $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ ✓

F é fechado!

MAS verdade que



$$\int_C F \cdot d\gamma = 2\pi \neq 0, \quad \forall R$$



Dado $F: \mathbb{R}^n \to \mathbb{R}^n$

Sabendo que F é gradiente

Como se determina o respectivo <u>potencial</u> escalar φ?

$F = \nabla \varphi$

$\varphi \equiv$ "fi"

<u>Exemplo</u>: $F(x,y) = (x,y)$; $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} = 0$

$$\varphi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C \text{ (constante)}$$

$$\nabla \varphi(x,y) = (x, y) \quad \checkmark$$

Exemplo: $F(x,y) = (y, x)$; $\frac{\partial F_1}{\partial y} = 1$; $\frac{\partial F_2}{\partial x} = 1$

$$\varphi(x,y) = xy + C$$

$$\nabla\varphi(x,y) = (y, x) \quad \checkmark$$

Exemplo: $F(x,y,z) = (x^2, y^3, z^4)$

$$\frac{\partial F_1}{\partial z} = 0 \qquad \frac{\partial F_3}{\partial x} = 0$$

$$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_2}{\partial x} = 0 \quad \checkmark$$ F é fechado

$$\frac{\partial F_2}{\partial z} = 0 \qquad \frac{\partial F_3}{\partial y} = 0$$

$$\varphi(x,y,z) = \frac{x^3}{3} + \frac{y^4}{4} + \frac{z^5}{5} + C$$

$$\nabla\varphi = F \quad \checkmark$$

Exemplo: $F(x,y,z) = (f(x), g(y), h(z))$

F é fechado.

$$\varphi(x,y,z) = \underbrace{\int f(x)}_{} + \underbrace{\int g(y)}_{} + \underbrace{\int h}_{} + C$$

primitiva

Exemplo: $F(x,y) = (-y, x)$

$$\frac{\partial F_1}{\partial y} = -1 \,, \qquad \frac{\partial F_2}{\partial x} = 1$$

F não é fechado $\Rightarrow$

F não é gradiente

F não tem potencial escalar!

Determinar $\boxed{\varphi}$ dado $\underset{\text{fechado}}{F}$: $\boxed{F = \nabla\varphi}$

$\underline{\mathbb{R}^2}$: $F : \mathbb{R}^2 \to \mathbb{R}^2$; $(F_1, F_2) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)$

$$\begin{cases} \frac{\partial\varphi}{\partial x}(x,y) = F_1(x,y) \\ \frac{\partial\varphi}{\partial y}(x,y) = F_2(x,y) \end{cases}$$

F_1, F_2 dados

$\downarrow$

$\varphi = ?$

$$\begin{cases} \frac{\partial\varphi}{\partial x} = F_1 \xrightarrow{\text{primitiva em } x} \varphi(x,y) = \int F_1(x,y)\,\underline{\underline{dx}} + \underbrace{A(y)}_{\text{constante em } x} \\ \frac{\partial\varphi}{\partial y} = F_2 \longrightarrow \ldots \end{cases}$$

$$\frac{\partial}{\partial y} \int F_1(x,y)\,dx + \boxed{A'(y)} = F_2$$

... etc

Exemplo: $F(x,y) = (x,y)$

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = x \longrightarrow \varphi(x,y) = \dfrac{x^2}{2} + \boxed{A(y)} \\ \dfrac{\partial\varphi}{\partial y} = y \longrightarrow 0 + \boxed{A'(y) = y} \end{cases}$$

$$A(y) = \frac{y^2}{2} + C$$

$$\Rightarrow \boxed{\varphi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C}$$

Exemplo: $F(x,y,z) = (2xyz, x^2z, x^2y)$

$$\varphi(x,y,z) = x^2yz + C$$

$$\frac{\partial \varphi}{\partial x} = 2xyz \longrightarrow \varphi(x,y,z) = x^2yz + \boxed{A(y,z)}$$

$B(z)$

$$\frac{\partial \varphi}{\partial y} = x^2z \longrightarrow \cancel{x^2z} + \frac{\partial A}{\partial y} = \cancel{x^2z}$$

$$\frac{\partial \varphi}{\partial z} = x^2y$$

$$\frac{\partial A}{\partial y} = 0$$

$$A(y,z) = B(z)$$

$$\cancel{x^2y} + B'(z) = \cancel{x^2y}$$

$$B'(z) = 0$$

$$B(z) = C$$

$$\boxed{\varphi(x,y,z) = x^2yz + C}$$